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## LETTER TO THE EDITOR

### Comparison of bootstrap percolation models

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**Abstract.** New simulations involving up to nearly  $10^9$  sites for square lattice bootstrap percolation give a percolation threshold compatible with unity. In this model sites in a square lattice are initially occupied randomly, and then are removed if they have less than three occupied neighbours. The results are compared with those of similar models, such as that studied by Frobose, and the earlier bootstrap and diffusion percolation models, in two, three and four dimensions, of Adler and Aharony. In the latter models, the effective threshold approaches its asymptotic limit (one or zero) as  $(\lambda/\ln L)^{d-1}$ , for samples of size  $L^d$ , in agreement with theoretical predictions. The coefficient  $\lambda$  decreases monotonically with the parameters which facilitate the removal (addition) of sites.

In bootstrap percolation (BP) [1] one starts with a random distribution of occupied (with probability  $p$ ) sites and then culls all those sites which do not fulfil a certain condition. For example, on a hypercubic  $d$ -dimensional lattice with  $2d$  nearest neighbours we may remove all those sites which do not have at least  $m = 2d - 1$  occupied neighbours. This irreversible removal of sites is continued until either the whole lattice has become empty, or until a stable configuration, where no more sites can be removed, is reached. There are several variants of this model, and numerous independent routes to its discovery. A summary of early work prior to 1986 is given by Adler and Aharony [2, 3], who classified the different variants of the model according to the types of percolation transition that occur. Adler and Aharony give exact results and bounds for the thresholds and present a mapping from bootstrap percolation, with concentration  $p$ , to diffusion percolation (DP), with concentration  $1 - p$ , models. In the latter, sites become occupied if a certain configuration of neighbouring sites is already occupied. The DP model that is equivalent to the specific case of bootstrap percolation with  $m = 2d - 1$  (which is referred to as BP throughout this letter) is called a  $2n$  DP in [2] and [3] (i.e. any two neighbours). This is the case where a site becomes occupied if any two of its nearest neighbours are occupied. Another DP model (called  $s2n$  DP), is one where a site becomes occupied if two of its neighbours that are second neighbours of each other, are occupied. The  $o2n$  DP model describes the case where a site becomes occupied if two of its neighbours that are opposite to each other are occupied. The occupation processes terminate when no more sites can be added. (For ease of comparison, unless specific mention of DP is made, we translate all general statements and numerical results below this point into the bootstrap representation of removing occupied sites; hence we consider thresholds near unity.)

The early bootstrap work was motivated by problems in the study of magnetic alloys [1]. Diffusion percolation was developed to model crack development in rocks weakened by fluid flow [2, 3]. Bootstrap models have also been applied to the study of orientational order in ortho-para hydrogen mixtures [4], and to the dynamics of the glass transition [5, 6]. Lenormand and Zarcone [7] developed a bond percolation model for flow in porous media, which can be related to BP/DP models.

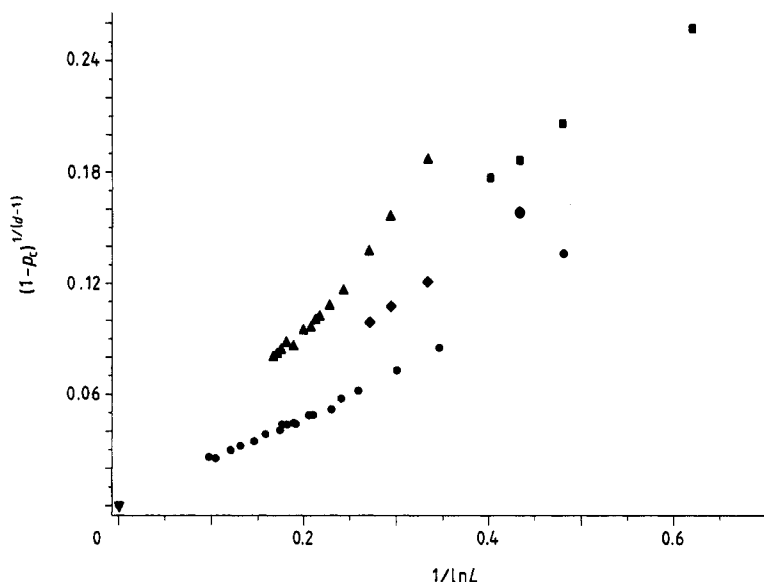
For BP on a finite lattice the stable configuration is a single spanning cluster of occupied sites and vacant rectangular holes. According to the 'void instability' argument of Straley, cited in [1] and some simulations [5] we expect a threshold concentration  $p_c$  of unity in an infinite system, i.e. even a minute concentration  $1-p$  of holes leads in a sufficiently large lattice to a totally empty lattice. For DP the threshold should be zero [2, 3], for those cases with 'cluster instabilities'.

The expected threshold  $p_c$  of unity (zero) is an exact result [1, 7, 8]. We expect  $p_c=0$  for s2n DP. Ertel *et al* [6], found a unit threshold in their diffusion model. However [2],  $p_c>0$  for o2n DP, and  $p_c<1.0$  for the models studied in [4], and for  $m=4$  bootstrap percolation on the simple cubic lattice [1]. Lenormand and Zarcone [7] suggested that for their two-dimensional model,  $1-p_c$  approaches zero only as  $\lambda/\ln L$  in an  $L\times L$  lattice. For their simulations  $\lambda$  was not small, and therefore they suggested that the approach of the threshold to unity [7] will be observable *only* for experimentally irrelevant large lattice sizes  $L$ . Nakanishi and Takano [5] empirically fitted their BP data to a logarithmic law, showing that the effect can be observed in experimentally accessible systems. Very recently, Aizenman and Lebowitz [9] bypassed most of the above studies and independently proposed the  $d$ -dimensional logarithmic law that  $1-p_c$  approaches zero only as  $(\lambda/\ln L)^{d-1}$  in an  $L^d$  lattice. They were able to obtain bounds on the prefactor  $\lambda$ .

In order to clarify the relationship between system size, model details and the prefactor we have simulated BP for larger samples (up to  $L=28\,800$ ) than previously considered. We reanalysed existing data for a variety of other BP and DP models [3], as well as our new results, to determine whether they are compatible with this logarithmic law and, if so, what the prefactor is. In particular we looked for systematic trends in the thresholds and prefactors of the related models. This is with the hope of explaining the recent results of Frobose [10] who found  $p_c$  up to 0.924 for a different bootstrap percolation problem on lattices of up to  $L=20\,032$ . Using a linear fit for  $p_c$  against  $1/\ln L$  he was unable to see a convergence of  $p_c$  to unity even for very large lattices, despite explicit exact predictions that this should also be true [6]. As we show,  $\lambda$  is sufficiently small for BP and s2n DP that we were able to observe the expected behaviour even for relatively small samples in two, three and four dimensions. This is the first time that this behaviour has been seen for any system except 2D BP [5].

Our new simulations for the BP case on the square lattice (2D) were done using several hours of Cray XMP/416, with one site per bit logical operations [6, 11] and about 700 updates per microsecond and processor. Smaller lattices had been run [3] with simpler techniques on an IBM 3090. For the largest lattice,  $L=28\,800$ , we made consecutive runs with decreasing  $p$  but using the same random numbers, until the system no longer settled into a stable configuration of rectangular holes. For smaller  $L$ , many such series of simulations were repeated.

We plot our new BP results in figure 1, together with data from [3]. The statistical error bars are of the order of the symbol size in figure 1. Assuming a linear variation of the effective thresholds  $p_c(L)$  with  $1/\ln L$  we extrapolate  $p_c$  to  $0.998\pm 0.004$  for infinite  $L$ , consistent with unity, for 2D, BP. Figure 1 also contains data from [3] for



**Figure 1.** Graph of  $(1 - p_c)^{1/(d-1)}$  as a function of  $1/\ln L$  for several models. The coefficient  $\lambda$  is the slope of this graph. The data are indicated by the symbols ●, ◆ and ■ for BP in two, three and four dimensions, respectively, and by ▲ for s2n DP in two dimensions. The symbol ▼ indicates the asymptotic result of  $L = \infty$  and  $p_c = 1.0$ .

those models where void instabilities can be expected: s2n DP in 2D and BP on the cubic and  $d = 4$  hypercubic lattices. These other models give similar  $p_c$  estimates, all of which are consistent with unity. Assuming  $p_c(\infty) = 1$ , our data are consistent with

$$(1 - p_c(L))^{1/(d-1)} = \lambda / \ln L + \dots \quad (1)$$

Our estimates of  $\lambda$  are given in table 1. We note that we were unable to fit (1) to the data from the cases where no void instabilities are expected, like the simple cubic lattice if sites with less than four neighbours are removed.

We now compare the various models systematically. The Frobose [10] model can be classified as a c3n DP model. As in s2n DP two neighbours on the same side must be occupied; but in addition the corner site between these two neighbours must also be occupied for the considered site to become occupied. This 'corner-three-neighbours' model c3n involving nearest and next-nearest neighbours is thus the most difficult of

**Table 1.** Bounds and estimates for  $\lambda$ .

Dimension	Model	Lower bound	Estimate	Upper bound
2	BP <sup>a</sup>	0.120	$0.245 \pm 0.015$	3.29
2	BP <sup>b</sup>	0.120	0.27	3.29
2	s2n DP <sup>c</sup>	0.480	$0.47 \pm 0.02$	3.29
3	BP <sup>c</sup>	0.271	$0.36 \pm 0.01$	6.93
4	BP <sup>c</sup>	0.442	$0.44 \pm 0.02$	12.63

<sup>a</sup> [3] and new data.

<sup>b</sup> [5].

<sup>c</sup> [3].

the three models to occupy. It is easy to see that the a2n model is occupied most easily, followed by s2n and then c3n. We expect (1) to be valid also for c3n. Thus  $p_c(L)$  should decrease and  $\lambda$  should increase if we go from a2n to s2n and then to c3n. In figure 1 we have plotted  $1-p_c$ , and we see that for a given  $L$ ,  $1-p_c$  increases as we go from a2n to s2n. Specifically, we fit the results for s2n with (1) and  $\lambda = 0.47 \pm 0.02$  and find that c3n must have  $\lambda > 0.8$ . Table 1 shows that  $\lambda$  can be quite small; however, it is nearly doubled from a2n to s2n and thus it might be doubled again if we go from s2n to c3n. Thus it is conceivable, as speculated by Frobose, that indeed at some very large  $L$ , perhaps beyond experimental relevance, a crossover to a logarithmic variation towards unity will happen also for the c3n case, since also then  $p_c(L = \infty) = 1$  according to [6].

Other BP models, for which we also expect (1) to be valid, unfortunately do not necessarily fit into this scheme of inequalities. The bond bootstrap percolation model (b2n) of Lenormand and Zircono [7] removes a site if two bonds at a 90 degree angle are empty; it then also removes the other two of the four bonds emanating from that site. In reversible  $m = 3$  bootstrap percolation, a site is emptied if it has less than three occupied neighbours (as in a2n) but is occupied again if it has three or more occupied neighbours. While its percolation threshold must be smaller than that for a2n it is not clear how it relates to c3n and is thus not included in our figure ( $p_c$  is about 0.9 for  $L = 2112$ ).

Aizenman and Lebowitz [9] also obtained bounds on the coefficient  $\lambda$ . Correcting an error in their equations (3.5) and (3.8), and evaluating their upper bounds, these are  $C_1 \leq \lambda \leq C_2$ , with

$$\begin{aligned} C_1 &= D^{-d/(d-1)} \sup\{g(z) | z \geq 0\} \\ C_2 &= d \int_0^\infty dz \frac{g(z)}{z} \\ &= \frac{d}{d-1} \zeta\left(\frac{d}{d-1}\right) \Gamma\left(\frac{1}{d-1}\right) \end{aligned} \quad (2)$$

where  $g(z) = -z \ln[1 - \exp(-z^{d-1})]$ , and  $\zeta(x)$  is the Riemann zeta function. Note that our a2n model (or equivalently the BP models) correspond to  $D = 2$  of [9] while our s2n model corresponds to  $D = 1$ . Table 1 also contains these bounds, and we note that our numerical values are quite close to the lower bounds.

Equation (1) was first derived heuristically by Lenormand and Zircono [7], for 2D. Their argument can easily be generalised to higher dimensions: a cubic void of size  $n^d$  will stop growing with probability  $P^-(n) \sim (1-p)^{2dn^{d-1}} \approx 1 - 2dpn^{d-1}$ , so the probability to grow is  $P^+(n) \approx 2dpn^{d-1}$ . At the threshold,  $P^+(n_c) \sim 2da$ , with  $a = O(1)$ , hence  $p_c \sim an_c^{-(d-1)}$ . The probability  $P(n)$  to find an  $n^d$  void in an  $L^d$  network obeys  $P(n+1) \approx (2dpn^{d-1})^d P(n)$ , hence  $P(n) \approx (2dp)^{dn} (n!)^{d(d-1)}$ . Writing  $L^d P(n_c) = b$ , with  $b = O(1)$ , and using Stirling's formula, we find

$$\ln L = \frac{\lambda}{p_c^{1/(d-1)}} + \frac{1}{2} \ln p_c + A \quad (3)$$

with  $\lambda = a^{1/(d-1)}(d-1 - \ln 2da)$  and  $A = \frac{1}{2} \ln[b^{2/d} a^{-1} (2\pi)^{-(d-1)}]$ . The term  $\frac{1}{2} \ln p_c$ , as well as some constants, were missed in [7]. The term  $\frac{1}{2} \ln p_c$  is the leading correction to the asymptotic equation (1). Since we also expect other corrections (e.g. of order  $1/n_c$ , from the approximations for  $P(n_c)$ ) and since the heuristic argument ignores corrections due to non-cubic voids, we feel that there is no justification to compare

data to values of the constant  $A$ . We note, however, that our critical values of  $\lambda$  yield values of  $a$  of order unity.

In summary, we have shown that (1) agrees beautifully with most BP and DP simulations, with coefficients which are consistent with heuristic arguments and rigorous bounds. We have also been able to order some models according to their difficulty (or ease) to form voids. Future theoretical work should try to evaluate the coefficients  $\lambda$  exactly, and discuss the corrections to the leading behaviour.

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### References

- [1] Chalupa J, Leath P L and Reich G R 1981 *J. Phys. C: Solid State Phys.* **12** L31  
Pollak M and Riess I 1975 *Phys. Status Solidi b* **69** K15  
Kogut P M and Leath P L 1981 *J. Phys. C: Solid State Phys.* **12** 3187
- [2] Adler J and Aharony A 1986 *STATPHYS 16* Boston Abstracts
- [3] Adler J and Aharony A 1988 *J. Phys. A: Math. Gen.* **21** 1387
- [4] Adler J, Palmer R G and Meyer H M 1987 *Phys. Rev. Lett.* **58** 882
- [5] Nakanishi H and Takano H 1986 *Phys. Lett.* **115A** 117
- [6] Ertel W, Frobose K and Jackle J 1988 *J. Chem. Phys.* **88** 5027  
Lebowitz J L 1988 Private communication
- [7] Lenormand R and Zircon C 1984 *Kinetics of Aggregation and Gelation* ed F Family and D P Landau (Amsterdam: Elsevier) p 177
- [8] Van Enter A C D 1988 *J. Stat. Phys.* **48** 943
- [9] Aizenman M and Lebowitz J L 1988 *J. Stat. Phys.* **21** 3801
- [10] Frobose K 1989 *J. Stat. Phys.* in press
- [11] Herrmann H J 1986 *J. Stat. Phys.* **45** 145